A NUMERICAL METHOD FOR THE SHEAR STRESS DETERMINATION

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SUMMARY

A theory has been carried out on the shear stress, assuming that the shear force can generate both warping displacement and rigid vertical translation of the structural section.

Relations have been obtained for the normal and tangential stresses, based on the Vlasov hypotheses about the thin-walled beams, and developed in orthogonal curvilinear coordinates, what allows to account for the influence of the branches curvature.

A numerical method has been also developed, which modifies the stationary method elaborated by Hughes, assuming a cubic law for the warping function, according to the verified Poisson equation, and utilizing the expressions obtained for the displacement and the stress fields.

In order to verify the influence of the shear deflection, a procedure has been adopted, which is a numerical translation of the Neumann problem, and substitutes the condition of absence of rigid warping components, for the assumption of zero values of the warping function on the neutral axis, assumed by Hughes, as a numerical translation of the mixed Dirichlet-Neumann problem.

The application to a simplified structure considered by Hughes, shows the negligible influence of the shear deflection, and allows to change the Hughes method for the only order of the warping function.

1. INTRODUCTION

In the traditional shear theory, bending and shear are assumed each other independent, and the shear effect reduces only to the warping; the invariability of the shear force is also admitted, and, as a consequence, no warping effects are considered on the normal stress. The stress field is also reduced to the only two components $\sigma$ and $\tau$, the first one determined by a De Saint Venant procedure, the second one by a Vlasov procedure. The physical consideration of the shear influence on the vertical displacement and of the bending and shear connection, induces to a re-examination of the theory, devoted to a careful individuation of the stress and strain fields, which may be useful to assign the approximation margin of the traditional theory.

2. THE FUNDAMENTAL EQUATIONS FOR THE VERTICAL BENDING-SHEAR STRAIN

Assuming the global and local Cartesian frames sketched in fig.1, the mixed $P(x, \eta, \zeta)$ representation can be utilized for the ship points, which reduces to the global one $P(x, y, z)$, when the approximate relation $\zeta = z$ is added to the rigorous one: $\eta = y - \theta$ what implies the approximate coincidence of the x-axis with the beam-axis.

Adopting the fundamental Vlasov structural hypothesis of maintenance of the cross section contour, the only warping displacement of the section has to be added to the rigid one; then the relation $v(x, \eta, \zeta)$, $j = 0$ for the y-component of the displacement function $v(x, \eta, \zeta)$, has to be assumed in a vertical bending-shear stress, because of the nature of the warping - longitudinal displacement variable on the section - and of the rigid motion induced by the load - composed by a $\theta(x)$ rotation of the section about its $\eta$-axis, and a $w(x)$ translation - for small deformations - which allow to write the rotation displacement as $\theta(x) \times (P - G(x)) = \theta(x) \zeta I$ - the following expression is obtained for the $v(x, \eta, \zeta)$ function:

$$v(x, \eta, \zeta) = (\theta(x) \zeta + u_v(x, \eta, \zeta) i) + w(x) j k$$

subjecting the $u_v(x, \eta, \zeta)$ function to a by parts decomposition, furnishes:

$$u_v(x, \eta, \zeta) = f(x) \phi(\eta, \zeta)$$

or also:

$$u_v(x, \eta, \zeta) = g(x) G_{1} \varphi(\eta, \zeta)$$

where the following positions have been assumed:

$$\begin{cases}
G = \text{Tangential modulus (first Lamé constant)} \\
I = \text{Inertia moment of the section about its } \eta \text{ axis}
\end{cases}$$
and the I moment has been supposed to be constant with 
x, according to the hypothesis of cylindrical hull –
approximately valid in the neighbourhood of the
section.
From (1) the strain components (for small deformation)
can be so derived :

\[
\begin{align*}
\varepsilon_y &= \varepsilon_y = \gamma_{xx} = 0 \\
\varepsilon_x &= e_x = \frac{d\theta(x)}{dx} + \frac{1}{2G_1} \frac{d}{dx} \frac{g(\eta, \zeta)}{\partial \eta} + \lambda(x) \\
\gamma_{xy} &= \frac{g(x)}{G_1} \frac{\partial \varphi(\eta, \zeta)}{\partial \eta}; \gamma_{yx} = \frac{g(x)}{G_1} \frac{\partial \varphi(\eta, \zeta)}{\partial \zeta} + \lambda(x)
\end{align*}
\]

when the \( \lambda(x) \) function is introduced by the position:

\[
\lambda(x) = \theta(x) + \frac{d w(x)}{dx}
\]  

The first (4.1) relations ,related to their dual ones of the
DSV (De Saint Venant) theory \( \sigma_y = \sigma_x = \tau_y = 0 \) (and
to the Poisson implications of the first two, for isotropic
linear elasticity: \( \varepsilon_y = \varepsilon_x = -v\varepsilon_x \), with: \( v = \) Poisson ratio
),show the extremely different behaviour assigned by the
Vlasov and the DSV theories to the lateral contraction of the
elementary longitudinal fibres of the structure, and
then of the cross structural section (restrained in the first
theory, free in the second one) and , consequently, the
extremely different role assigned, in the primary stress, to
the transverse frames and bulkheads , assumed rigid in the
first theory , without any strength in the second one.

3. EXPRESSIONS OF THE STRAIN COMPONENTS IN ORTHOGONAL CURVILINEAR COORDINATES

Substituting the bulb sections with the equivalent angle
profiles (with the net thickness of the web equal to that
of one of the bulb section , as in the RINA rules, and
the other three dimensions obtained, imposing equal values
of the net areas and inertia moments , and equal centre
positions) allows to consider the structural section
constituted by branches of constant thickness , as that
one sketched in fig.2.
In each branch, three parallel curves can be individuated:
the contours \( l_1 \) and \( l_2 \) lying on the structure boundary
and the centre line \( l \); then on the centre line \( \ell \) a
curvilinear abscissa \( s \) can be instituted , with the

\[
\text{Fig. 2}
\]

origin in either extremity (node) of the line; finally the
branch can be referred to the orthogonal curvilinear
coordinates \((s, n)\), with \( s \) the abscissa on \( \ell \) of the

thickness line through the considered point \( P \), \( n \) the
linear abscissa on the P thickness line, with origin on \( \ell \).
Denoting by \( \mathbf{r} \) the position vector relative to \( O \) gives, for
the orthogonal coordinate curves through the point
\( P(\pi, \tau) \),the vectorial equations:

\[
\begin{align*}
\mathbf{r} &= \mathbf{r}(s, \pi) \\
\mathbf{r} &= \mathbf{r}(\pi, n)
\end{align*}
\]

the first one then coincides with the parallel line to \( \ell \),
the second one with the \( P \) thickness line; the "natural"
basics , variable with \( P \), will be then the system of the two
orthogonal unit vectors :

\[
\begin{align*}
\frac{d \mathbf{r}}{ds} \times \frac{d \mathbf{r}}{dn}
\end{align*}
\]

Both basis vectors are constant on any thickness line;
their values then coincide with those ones assumed in the
\( \ell \) intersection ,and can be expressed by the functions
of the only \( s \) variable:

\[
t(s) = \frac{d \mathbf{r}}{ds}; \quad \mathbf{n}(s) = t(s) \times \mathbf{i}
\]

when the conventional position has been assumed for the
\( \ell \) equation: \( \mathbf{r}(s, 0) = \mathbf{r}(s) \), and the \( n \) coordinate sign has
been assumed , accordingly to the vector product (8).
Denoting by \( \varphi(s,n) \) the composite function of the three
ones : \( \varphi(\eta, \zeta); \eta(s,n), \zeta(s,n) \) and utilizing the (1), (3),
(8) relations, give for the displacement function ,the
expression:

\[
v(s,s,n) = \left( \theta(s) \zeta(s,n) + \frac{\varphi(s,n)}{G_1} \right) i + w(s) \left( \frac{d \zeta}{ds} t(s) - \frac{dn}{ds} n(s) \right)
\]

Let’s now introduce , as in the classical Love’s treatment,
the functions \( h_p(s,n) \), with \( p = s, n \), connected with the
Lamé parameters \( \lambda_p(s,n) \) by the relations:

\[
\begin{align*}
h_p(s,n) &= 1/h^*_p(s,n) \quad \text{and so given by:}
\end{align*}
\]

\[
\begin{align*}
h_s(s,n) &= \frac{\partial \mathbf{P}}{\partial s}; \\
h_n(s,n) &= \frac{\partial \mathbf{P}}{\partial n}
\end{align*}
\]

let \( \rho(s,n) \) and \( \rho(s) = \rho(s,0) \) be the algebraic curvature
radii of the \( \mathbf{r} = \mathbf{r}(s,n) \) and \( \mathbf{r} = \mathbf{r}(s,0) \) lines:

\[
\begin{align*}
\rho(s) &= (C(s) - P(s)) \times n(s) \\
\rho(s,n) &= (C(s) - P(s,n)) \times n(s) = \rho(s) - n
\end{align*}
\]

With: \( C(s) \) = curvature centre of the parallel lines
through the \( s \) thickness line.
The two relations:
\begin{align*}
\{P(s) - C(s)\} &= -p(s)n(s) \quad (12) \\
\{P(s, n) - C(s)\} &= -\rho(s) n(s) \quad (13)
\end{align*}

will be then valid, and the first one differentiation preliminarily furnishes, by the second Frénet formula, the relation:

\begin{align*}
t(s) - \frac{dC}{ds} = -\rho(s)n(s) + t(s) = \frac{dC}{ds} = \frac{dP}{ds}n(s)
\end{align*}

which allows to simplify the first P partial derivative, and to obtain:

\begin{align*}
\frac{\partial P}{\partial s} = (\rho(s) - n)(t(s)) \\
\frac{\partial P}{\partial n} = n(s)
\end{align*}

Finally, being essentially positive the \(\rho(s) - n\)/\(\rho(s)\), because of the thickness negligibility compared with the \(\rho(s)\) radius, gives:

\begin{align*}
h(s, n) = \frac{\rho(s)}{\rho(s) - n} \\
h(s, n) = 1
\end{align*}

It is possible to write the six strain components referred to the curvilinear coordinates \((x, n, s)\), with “natural” basis the system \(\left\{ \frac{\partial}{\partial n}, \frac{\partial}{\partial s} \right\}\), and obviously \(h_x = 1\):

starting from their general expressions in orthogonal curvilinear coordinates:

\begin{align*}
\varepsilon_p &= h_p \frac{\partial u_p}{\partial \alpha_p} + \sum_{r \neq p} h_r \frac{\partial u_r}{\partial \alpha_p}, \\
\gamma_{pq} &= h_q \frac{\partial (h_u u_p)}{\partial \alpha_p} + h_u \frac{\partial (h_u u_p)}{\partial \alpha_q}
\end{align*}

with:

\begin{align*}
\alpha_p &= x, n, s \\
u_p &= v \cdot e_p \\
e_p &= i, n, t
\end{align*}

and utilizing the \((8), (15)\) relations, the obvious one \(\frac{\partial}{\partial n} \frac{dn}{ds} = 0\) and the first Frené formula—which implies:

\begin{align*}
\frac{d^2 \zeta}{ds^2} = -\frac{1}{\rho(s)} \frac{dn}{ds}, \quad \frac{d^2 n}{ds^2} = \frac{1}{\rho(s)} \frac{d \zeta}{ds}
\end{align*}

the following expressions can be easily obtained:

\begin{align*}
\varepsilon_s &= \varepsilon_n = \gamma_{ss} = 0 \\
\varepsilon_x &= \frac{d \theta(x)}{dx} \zeta(x, s, n) + \frac{1}{G} \frac{d^2 g(x)}{dx^2} \varphi(s, n) \\
\gamma_{ss} &= \lambda(x) \frac{d \zeta}{ds} + \frac{\rho(s)}{\rho(s) - n} g(x) \frac{\varphi(s, n)}{G} \\
\gamma_{ss} &= -\lambda(x) \frac{d n}{ds} + g(x) \frac{\varphi(s, n)}{G}
\end{align*}

All the \((17)\) relations can be also obtained, starting from the \((4)\) ones, and applying the relation:

\begin{align*}
\gamma_{pq} = (E e_p) - e_q
\end{align*}

with: \(E = \) strain tensor; \(\gamma_{pq} = \varepsilon_p\).

The \(\gamma_{ss}\) expression shows the influence of the branch curvature on the strain; this influence then vanishes for straight branches—as those ones considered in the numerical method introduced in the paper—for which the limit condition \(\rho(s) - n\)/\(\rho(s)\) \(\rightarrow 1\) implies:

\begin{align*}
\gamma_{ss} &= \lambda(x) \frac{d \zeta}{ds} + g(x) \frac{\varphi(s, n)}{G}
\end{align*}

4. THE VERTICAL BENDING-SHEAR STRESS

Referring to the “natural” basis \(\left\{ \frac{\partial}{\partial n}, \frac{\partial}{\partial s} \right\}\) allows to directly utilize the approximate anisotropic behaviour of a thin plate: rigid along the thickness, elastic along the orthogonal direction; what implies that the only elastic stress components are those ones: \(\sigma_x, \sigma_y, \tau_{xy}\) and, consequently, that both in the Navier relations, and in the expression of the Beltrami-von Mises sigma, these ones will be only involved.

As a consequence of above, the following relations will hold:

\begin{align*}
\sigma_x &= \frac{E}{(1 - \nu^2)} \varepsilon_x = \frac{E}{(1 - \nu^2)} \left( \frac{d \theta}{dx} \zeta(x, n) + \frac{1}{G} \frac{d^2 g(x)}{dx^2} \varphi(s, n) \right) \\
\sigma_y &= \nu \sigma_x \\
\tau_{xy} &= G \lambda(x) \frac{d \zeta}{ds} + g(x) \frac{\varphi(s, n)}{G}
\end{align*}

\begin{align*}
\sigma = \sqrt{\left( \frac{1 - \nu + \nu^2}{(1 - \nu^2)^2} \right) E^2 \varepsilon_x^2 + 3 \tau_{xy}^2}
\end{align*}

with: \(E=\) Young modulus, \(\nu=\) Poisson ratio; \(\sigma=\) Beltrami-von Mises sigma.

The second one reduces, for steels (\(\nu=0.3\)), to \(\sigma = \sqrt{0.954 E^2 \varepsilon_x^2 + 3 \tau_{xy}^2}\), which is lightly lower than the value \(\sqrt{E^2 \varepsilon_x^2 + 3 \tau_{xy}^2}\) obtained, taking \(\sigma_x = E \varepsilon_x\), as it is
currently made; then the current procedure is lightly in
favour of the safety.
As regards the unknown functions
g(x), \theta(x), \lambda(x), \varphi(s,n), let’s start from the indefinite and
boundary equilibrium equations, which naturally involve
all the stress components; the first one can be rewrite,
involving the unit surface load \( p \), and neglecting the
body forces: because of the thickness smallness, it can
be obtained by a direct differential procedure— as that
one showed in Fig.3 for the x-component — applied to an
infinitesimal volume element, whose one elementary
infinitesimal volume element, whose one elementary
section, allows to introduce
the unit vertical load, also give the
function, to develop the first (23) equation,
and that one of the bending moment
expression of

\[ g(x) = \frac{1}{I} \int_{A(x)} \nabla \varphi \cdot k dA + G \lambda(x) A(x) \]  

which can be rewritten — or also directly derived from
(17) — in the form:

\[ T(x) = \frac{g(x)}{I} \sum_{i=1}^{N} \int_{A_i} \left( \frac{\partial \varphi}{\partial s} \frac{d \xi}{d s} - \frac{\partial \varphi}{\partial n} \frac{d \zeta}{d s} \right) dA + G \lambda(x) A(x) \]  

when all the \( N \) branches \( A_i \) (s) are straight.

An only one equation has been so obtained with two
unknown \( x \)-functions — for given section geometry, and
load distribution — \( g(x) \) and \( \lambda(x) \); one of them ca be
then arbitrarily assigned, the other one will be consequently determined; according to the usual
procedure, it shall be assumed:

\[ g(x) = T(x) \]  

so the dimensional equation \( [\varphi] = [L] \) is verified, and
\( \lambda(x) \) can be expressed by the shear force, as:

\[ \lambda(x) \frac{G A(x)}{T(x)} = 1 - \sum_{i=1}^{N} \int_{A_i} \left( \frac{\partial \varphi}{\partial s} \frac{d \xi}{d s} - \frac{\partial \varphi}{\partial n} \frac{d \zeta}{d s} \right) dA \]  

The other \( x \)-function \( \theta(x) \) can be also expressed by the
loads: starting from the new expression of \( \sigma_x \):

\[ \sigma_x = \frac{E}{I} \left( \frac{d \theta}{d x} \xi(s,n) + \frac{c(x)}{G I} \varphi(s,n) \right) \]  

and that one of the bending moment \( M(x) \):

\[ M(x) = \int_{A(x)} \sigma_x \xi dA = \frac{E I}{I} \left( \frac{d \theta}{d x} \xi(s,n) + \frac{c(x)}{G I} \varphi(s,n) \right) \int_{A(x)} \phi \xi dA \]  

It is convenient, because of the remarkable properties of the \( \varphi(\eta, \zeta) \) function, to develop the first (23) equation,
in \( (\eta, \zeta) \) coordinates: from (4) it then follows:

\[ \frac{g(x)}{l} \nabla^2 \varphi = - \frac{E}{l^2} \left( \frac{d^2 \theta(x)}{dx^2} + \frac{1}{G I} \frac{d^2 g(x)}{dx^2} \right) \]  

Let’s now consider that the symmetry of the structural
section \( A(x) \), respect to the \( \zeta \) axis, allows to introduce
the parity notion, respect to \( \eta \), for functions defined on
\( A(x) \); then the function \( g(x) \) has to be even, because that one \(- \frac{E}{l^2} \frac{d^2 \theta(x)}{dx^2} \) is even, what is satisfied, if \( \varphi \) is, in its turn, even , as we
shall from now on suppose.
Utilizing (4) and denoting by \( T(x) \) the shear force, by
\( c(x) = dT/dx \) the unit vertical load, also give the equation:

\[ T(x) = \frac{g(x)}{l} \int_{A(x)} V \varphi k dA + G \lambda(x) A(x) \]  

Fig.3

where \( \tau \) denotes the tangential stress given by:

\[ \tau = \frac{g(x)}{l} \nabla \varphi + G \lambda(x) k \]  

The only interesting scalar equations, in a study of the
hull girder strength, are the x-projections of the vectorial
(22) ones — indeed the other ones include the \( p_x \) and
\( p_n \) components, that coincide with the transverse
stiffeners reactions, and can be, by them, determined —
which reduce, for \( p_x = 0 \) and \( n \cdot i = 0 \), according to the
hypothesis of cylindrical hull, to the following ones:

\[ \begin{cases}
\text{div} \Sigma + \frac{p}{l} = 0 \\
\Sigma n = p
\end{cases} \]  

(22)

with:

\[ \Sigma = \begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_y & 0 \\
\tau_{xz} & 0 & \sigma_z
\end{bmatrix} \text{ stress tensor} \]  

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hypothesis of cylindrical hull, to the following ones:

\[ \begin{cases}
\text{div} \tau = - \frac{\partial \sigma_x}{\partial x} \\
\tau \cdot n = 0
\end{cases} \]  

(23)

where \( \tau \) denotes the tangential stress given by:

\[ \tau = \frac{g(x)}{l} \nabla \varphi + G \lambda(x) k \]  

\[ \frac{g(x)}{l} \nabla^2 \varphi = - \frac{E}{l^2} \left( \frac{d^2 \theta(x)}{dx^2} + \frac{1}{G I} \frac{d^2 g(x)}{dx^2} \right) \]  

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Utilizing (4) and denoting by \( T(x) \) the shear force, by
\( c(x) = dT/dx \) the unit vertical load, also give the equation:

\[ T(x) = \frac{g(x)}{l} \int_{A(x)} V \varphi k dA + G \lambda(x) A(x) \]  

which can be rewritten — or also directly derived from
(17) — in the form:

\[ T(x) = \frac{g(x)}{l} \sum_{i=1}^{N} \int_{A_i} \left( \frac{\partial \varphi}{\partial s} \frac{d \xi}{d s} - \frac{\partial \varphi}{\partial n} \frac{d \zeta}{d s} \right) dA + G \lambda(x) A(x) \]  

when all the \( N \) branches \( A_i \) (s) are straight.

An only one equation has been so obtained with two
unknown \( x \)-functions — for given section geometry, and
load distribution — \( g(x) \) and \( \lambda(x) \); one of them ca be then arbitrarily assigned, the other one will be consequently determined; according to the usual
procedure, it shall be assumed:

\[ g(x) = T(x) \]  

so the dimensional equation \( [\varphi] = [L] \) is verified, and
\( \lambda(x) \) can be expressed by the shear force, as:

\[ \lambda(x) \frac{G A(x)}{T(x)} = 1 - \sum_{i=1}^{N} \int_{A_i} \left( \frac{\partial \varphi}{\partial s} \frac{d \xi}{d s} - \frac{\partial \varphi}{\partial n} \frac{d \zeta}{d s} \right) dA \]  

The other \( x \)-function \( \theta(x) \) can be also expressed by the
loads: starting from the new expression of \( \sigma_x \):

\[ \sigma_x = \frac{E}{l^2} \left( \frac{d \theta}{d x} \xi(s,n) + \frac{c(x)}{G I} \varphi(s,n) \right) \]  

and that one of the bending moment \( M(x) \):

\[ M(x) = \int_{A(x)} \sigma_x \xi dA = \frac{E I}{l^2} \left( \frac{d \theta}{d x} \xi(s,n) + \frac{c(x)}{G I} \varphi(s,n) \right) \int_{A(x)} \varphi \xi dA \]
the bending-shear equation can be derived in the form:

\[
\frac{d\theta}{dx} = \left(1 - \nu^2\right)M(x) - \frac{c(x)}{EI} \int_{A(s)} \phi \xi dA
\]  
(31)

Then a new expression for \(\sigma_z\) can be carried out, utilizing (30) and (31):

\[
\sigma_z = \frac{M(x)}{I} \xi + \frac{2c(x)}{EI} \left[\phi - \xi \int_{A(s)} \phi \xi dA\right]
\]  
(32)

while the new expression of \(\tau\) is the following one:

\[
\tau = \frac{T(x)}{I} \left[\text{grad} \phi + G\lambda (x)k\right]
\]  
(33)

which furnishes, for straight branches:

\[
\begin{cases}
\tau_{xx} = Gn\lambda (x) \frac{d\zeta}{ds} + \frac{T(x)}{I} \frac{\partial \phi(s,n)}{\partial s} \\
\tau_{xn} = -Gn\lambda (x) \frac{d\eta}{ds} + \frac{T(x)}{I} \frac{\partial \phi(s,n)}{\partial n}
\end{cases}
\]  
(34)

Finally assuming \(\frac{d\xi}{dx} = 0\) – the still water bending moments and the vertical wave shear force proposed by the rules are linearly variable with \(x\) – gives, by equations (23), (32), (33):

\[
\begin{cases}
\nabla^2 \varphi = -\zeta \\
\frac{\partial \varphi}{\partial n} = G1\lambda (x) \alpha_n \\
\varphi = 0
\end{cases} \quad \forall P \in A(x)
\]  
(35)

Then \(\varphi\) appears the solution of a Neumann problem; so it is determinate except an arbitrary constant, and its determination can be only carried out by a step by step procedure, because of the dependence of the boundary condition on the function \(\lambda (x)\), which, in its turn, depends on \(\varphi\).

As regards the indeterminacy, it is generally removed, for monoconnected sections, assigning the \(\Phi\) value in the section centre; for multiconnected sections, whose centre doesn’t normally belong to the \(\varphi\) domain, a method, useful for the numerical applications, may consist of a separate determination of the two \(\Phi\) restrictions to the parts \(A_1(x)\) and \(A_2(x)\) of \(A(x)\), the first one above the neutral axis, the second one under it; each one uniquely determined as solution of the mixed Dirichlet-Neumann problem, given by the assumption \(\varphi=0\) on the neutral axis:

\[
\begin{cases}
\nabla^2 \varphi = -\zeta \\
\frac{\partial \varphi}{\partial n} = G1\lambda (x) \alpha_n \\
\varphi = 0
\end{cases} \quad \forall P \in A_i(x)
\]  
(36)

It is also interesting to note that the second equation (35) makes \(\varphi\) dependent on \(x\), what appears physically evident, but is in contrast with the initial decomposition of the \(u_i(x,\eta,\xi)\) function. A different theory could be carried out, starting from the position: \(u_i(x,\eta,\xi) = \frac{T(x)}{G1} \varphi(x,\eta,\xi)\); but it would give the same results, as those ones till now obtained, when the \(\partial \varphi/\partial x\) derivative was neglected, as it is practically necessary.

5. THE SHEAR INFLUENCE ON THE VERTICAL SECTION DISPLACEMENT

In the practical procedure, it is admitted that the vertical displacement \(w(x)\) of the beam section is the only one determined by the rotation, and then it is connected with the rotation \(\theta(x)\) by the geometrical condition of orthogonality between the section and the elastic curve. This property, valid for monoconnected sections, whose axis is a material line, and belongs to the domain of the displacement function, can be extended to the multiconnected sections, admitting the orthogonality between the section and the elastic surface – deformed shape of the neutral surface (intersection of the structure and the surface with the \(\zeta=0\) generatrices). Assuming the approximate relation \(\zeta=x\), and accounting for the (1), (2), (3) and (28) relations, allow to write the vectorial position function in the form:

\[
X = x + \left(\theta(x)z + \frac{T(x)}{G1} \varphi(y,x)\right)i + w(x)k
\]  
(37)

The vectorial equation of the elastic surface is then:

\[
X = x + \left(\frac{T(x)}{G1} \varphi(y,0)\right)i + w(x)k
\]  
(38)

and reduces to the other one:

\[
X = x + w(x)k \quad \text{with} \quad x = (x i + y j)
\]  

and normal directed as \(\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} = k - \frac{d w}{d x} i\) when the assumption: \(\varphi(y,0) = 0\) is made, as above. The equation of the rotate section is, on the other hand:

\[
X = x + \theta(x)z i + w(x)k \quad \text{with} \quad x = \text{const. and normal directed as} \frac{\partial X}{\partial y} \times \frac{\partial X}{\partial z} = i - \theta(x)k
\]  

then the orthogonality condition is verified if (and only if) is:

\[
\lambda(x) = \theta(x) + \frac{d w}{d x} = 0
\]  
(39)

So in the practical procedure, bending and shear are assumed each other independent, and the shear effect reduces only to the warping; the physical consideration of the shear influence on the vertical displacement and of
the bending and shear connection, induces to the introduction of the $\lambda(x)$ function.

6. THE NUMERICAL DETERMINATION OF THE SHEAR STRESS

The shear flow equation, applied to the branches, shows that the maximum tangential stress supported by the stiffeners is proportional to their first moment about the neutral section axis, and then is negligible, because of the moment smallness; but each stiffener causes a jump discontinuity, proportional to its first moment, in the plating branch; therefore no primary shear check is required for the stiffeners, but their contribution to the plating shear strength has to be considered; a simple way of accounting for the global shear contribution of the attached stiffeners to the plating branch is “spreading” them, by a thickness plating increment, able to give a first moment increment equal to the moment value $\sum m_i$ of the attached stiffeners:

$$l(\Delta t)\zeta = \sum_i m_i$$

(40)

with $l, \zeta =$ branch length and centre line coordinate

It is now possible to reduce the bidimensional Dirichlet-Neumann problems (36) – or Neumann problem (35) – to a monodimensional one, assuming all the geometrical and mechanical quantities constant on the thickness branch, with their integral mean values; what is rigorously verified by the unit vectors of the “natural” basis, and can be accepted for the other ones, because of the thickness smallness.

The $\zeta$ mean value coincides with the $\zeta(s)$ value on the centre line, as it immediately follows from the relation:

$$\left[ \int_{-\infty}^{\infty} \zeta(s,n) d n \right]^{1/2} = \frac{1}{\infty} \int_{-\infty}^{\infty} \frac{\zeta(s) - \zeta(s,t/2)}{t} \frac{d n}{\zeta(s)}$$

As regards the $\tau(x,s)$ mean value, it has been defined by:

$$\tau(x,s) = \int_{-\infty}^{\infty} \tau(x,s,n) d n$$

(41)

and can be also written in the form:

$$\tau(x,s) = \tau(x,s) \cdot \left[ \tau(x,s) \right] = \tau_{ss}, \tau_{ss} = \tau_{xx},$$

(42)

with $\tau(x,s)$ the extreme values $\tau(x,s,-t/2), \tau(x,s,t/2)$ of the integrand are orthogonal to the thickness, as the second (23) requires, and then all vector values can be assumed equally directed, as the thickness smallness allows.

In all the equations and relations, till now considered, the monodimensional parameter $s$ can be substituted for the bidimensional one $(s, n)$, and assuming all the branches straight – then approximating a curvilinear one by a sufficient number of straight branches with nodes on its centre line – what allows to express all the vector operators, in the same way they are for cartesian bases $(h_i = h_m = h_s = 1)$, the following relations particularly hold in each branch:

$$v(x,s) = \left( \theta(x) + \frac{T(x)}{G I} \right) \frac{\phi(s)}{\bar{n}} + \frac{d\zeta}{ds} - \frac{d \bar{n}}{ds} \frac{n}{n}$$

(43)

$$\sigma_s(x,s) = \frac{M(x)}{I} \frac{\phi(s)}{\bar{n}} + \frac{2c(x)}{(1-\nu)l} \int_0^l \phi(s) \frac{\zeta(s)}{1} \frac{d\zeta}{ds} \frac{d s}{ds}$$

(44)

$$\tau(x,s) = G \lambda(x) \frac{d\zeta}{ds} + \frac{T(x)}{I} \frac{d\phi(s)}{ds}$$

(45)

$$\frac{d^2\phi(s)}{ds^2} = -\zeta(s)$$

(46)

$$\zeta(s) = \zeta_{mi} + \frac{\zeta_{ni} - \zeta_{mi}}{l_i} s$$

(47)

with $l_i, t_i = \text{length and thickness of the } i\text{-th branch; } m < n = \text{nodal indices}$

The last two relations uniquely determine the $\phi(s)$ function and, consequently the $\frac{d\phi(s)}{ds}$ one, by the relations:

$$\phi(s) = \phi_{ni} + \left( \frac{\zeta_{ni} - \zeta_{mi}}{l_i} \right) \frac{l_i + (|\zeta_{ni} + 2\zeta_{ni}|)}{6} \left( \zeta_{mi} + \frac{(\zeta_{ni} - \zeta_{ni})}{2} \right)$$

$$\frac{d\phi(s)}{ds} = \frac{\phi_{ni} - \phi_{ni}}{l_i} \frac{l_i + (|\zeta_{ni} + 2\zeta_{ni}|)}{6} \left( \zeta_{mi} + \frac{(\zeta_{ni} - \zeta_{ni})}{2} \right)$$

(48)

The stress components can be then rewritten in the form:

$$\sigma_s(x,s) = \frac{M(x)}{l_i} \left( \zeta_{mi} + \frac{\zeta_{ni} - \zeta_{mi}}{l_i} s \right) + \frac{2c(x)}{(1-\nu)l} \psi(s)$$

(49)

with:

$$\psi(s) = \phi_{ni} + \left( \frac{\zeta_{ni} - \zeta_{mi}}{l_i} \right) \frac{l_i + (|\zeta_{ni} + 2\zeta_{ni}|)}{6} \left( \zeta_{mi} + \frac{(\zeta_{ni} - \zeta_{ni})}{2} \right)$$

$$- \kappa \left( \zeta_{mi} + \frac{\zeta_{ni} - \zeta_{mi}}{l_i} s \right)$$

(50)
which can also be reduced, neglecting $\lambda^2$ and the warping part of $\sigma$, to the other one:

$$
2\sum_{i=1}^{M} \int_{0}^{l_i} \left( \left[ \sigma_{x}(x,s) \right] \frac{d\sigma}{ds} + \left[ \sigma_{y}(x,s) \right] \frac{d\sigma}{ds} + \frac{1}{2} \left( \frac{d\sigma}{ds} \right)^2 \right) ds = 0
$$

(55)

The problem of the $\varphi(s)$ determination, then reduces to the variational one of the research of extrema of the functional $U(\varphi(s)) \frac{d\varphi}{ds}$, or also of the functional, the only part of $U$ – without an influence of the multiplicative constant – which depends on $\varphi(s)$ and $\frac{d\varphi}{ds}$:

$$
2\sum_{i=1}^{M} \int_{0}^{l_i} \left( \frac{1}{2} \left( \frac{d\varphi}{ds} \right)^2 + G_{2} \lambda(x) \frac{d\varphi}{ds} \right) ds = 0
$$

(56)

Introducing the expressions (48) reduces the functional to a function of the $\varphi$ nodal values, and the stationery condition, to the system of the $M$ linear equations (the uniform continuity of the integrand function, allows the derivation under integral sign):

$$
\sum_{i=1}^{n(k)} \frac{t_i}{l_i} \frac{d\varphi}{ds} \left( \frac{l_i}{2} \frac{d\varphi}{ds} \right) + G_{2} \lambda(x) \frac{d\varphi}{ds} \left( \frac{l_i}{2} \frac{d\varphi}{ds} \right) = 0
$$

(57)

For $k=1, \ldots, M$

with: $n(k)=\text{number of the branches concurrent in the k-th node}$

Denoting by $r$ the node different from the k-th one, for each branch concurrent in the k-th node, and neglecting the dependence of $\lambda$ on the $\varphi$ nodal values, give the system:

$$
\sum_{i=1}^{n(k)} \frac{t_i}{l_i} (\varphi_{ri} - \varphi_{ri}) = \frac{1}{6} \sum_{i=1}^{n(k)} \frac{t_i}{l_i} (2\varphi_{ri} + \varphi_{ri}) - \mu(x) \sum_{i=1}^{n(k)} \frac{t_i}{l_i} (\varphi_{ri} - \varphi_{ri})
$$

For $k=1, \ldots, M$

(58)

7. A NUMERICAL APPLICATION

In order to estimate the influence of the shear deflection (then of the last term in the (58) equation), an application has been carried out, based on the simplified structure considered by Hughes [1] (see Fig.4); concerning this, two numerical procedures can be developed: the first one, adopted by Hughes, is a numerical translation of the mixed Dirichlet-Neumann problem, and by the assumption $\varphi=0$ on the neutral axis, allows to operate on two systems with a reduced number of equations; but its application to the equations
that include nodal values lying to both parts $A_i(x)$ and $A_i(x)$ of $A(x)$—necessarily implies a step by step procedure (it is interesting to note that the reduction of the bidimensional Dirichlet-Neumann problem (36) to a monodimensional one, allows to reduce the second (36) to: $\partial \phi / \partial n = 0$, and so to make the problem direct, from a theoretical point of view).

The second one, is a numerical translation of the application).

A validation of this last procedure has been carried out , by a comparison with the results obtained by the Hughes−theory: neglecting the last term of the (36) equations according to the hypotheses of the shear flow theory— and denoting by $q(s)$ the normalized shear flow, the difference $\phi_{i+1} - \phi_{m+1}$ obtained by the Hughes method—which just utilizes the (36) equations without the last term—and that one obtained by the relation

$$\int_0^l \phi(s) ds = 0$$

for anyone of the (36) system (the first one in our application).

A validation of this last procedure has been carried out, by a comparison with the results obtained by the flow theory: neglecting the last term of the (36) equations— according to the hypotheses of the shear flow theory—and denoting by $q(s)$ the normalized shear flow, the difference $\phi_{i+1} - \phi_{m+1}$ obtained by the Hughes method— which just utilizes the (36) equations without the last term— and that one obtained by the relation

$$\frac{1}{t_i} \int_0^l q(s) ds$$

have been compared, with the same results.

![Fig.4](image)

The system (36)— written without any attention to the nodes numbering, because of the small rank of the coefficients matrix, and simplified dividing by the $t_i = t_4$ thickness— reduces to the matrix equations (the first one without, the second one with the terms, accounting for the shear deflection):

\[
\begin{bmatrix}
10 & 40 & 30 & 41.25 & 60 & 18.75 & \phi_1 \\
-0.1 & 0.25 & -0.1 & 0 & -0.05 & 0 & \phi_2 \\
0 & -0.1 & 0.15 & -0.05 & 0 & 0 & \phi_3 \\
0 & 0 & -0.05 & 0.2625 & -0.2125 & 0 & \phi_4 \\
0 & -0.05 & 0 & -0.2125 & 0.45 & -0.1875 & \phi_5 \\
0 & 0 & 0 & 0 & -0.1875 & 0.1875 & \phi_6
\end{bmatrix} = \begin{bmatrix}
-12s \\
-137.42 - 12s + 0.5s^2 \\
17.42 - 12s \\
-102.60 - 12s + 0.5s^2 \\
-67.10 + 8s \\
-80.00 + 8s
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
10 & 40 & 30 & 41.25 & 60 & 18.75 & \phi_1 \\
-0.1 & 0.24 & -0.11 & 0.01 & -0.04 & 0 & \phi_2 \\
0 & -0.11 & 0.14 & -0.04 & 0.01 & 0 & \phi_3 \\
0 & 0.01 & -0.04 & 0.2531 & -0.2219 & 0 & \phi_4 \\
0 & -0.04 & 0.01 & -0.2219 & 0.44 & -0.1875 & \phi_5 \\
0 & 0 & 0 & 0 & -0.1875 & 0.1875 & \phi_6
\end{bmatrix} = \begin{bmatrix}
-12s \\
-137.42 - 12s + 0.5s^2 \\
17.42 - 12s \\
-102.60 - 12s + 0.5s^2 \\
-67.10 + 8s \\
-80.00 + 8s
\end{bmatrix}
\]

so obtaining as maxima absolute values $|\tau_3|$ (resp. $|\tau_4|$) of the normalized stresses on the intersection between the neutral axis, and the second (resp. the fourth) branch, those ones (see Fig. 5):

$$|\tau_3| = 209.42 \text{ m}^2 \quad |\tau_4| = 174.60 \text{ m}^2$$

![Fig.5](image)
These results have been obtained by developing a dedicated program in MATLAB that allows to calculate the shear flows normalized stresses for a generic section, both applying the classical theory or the modified one, only by introducing its geometry. In this program it is also possible to impose the condition $\Phi = 0$ on the neutral axis or the other one $\int_{A(s)} \varphi \, dA = 0$.

8. CONCLUSIONS

The numerical application shows that the influence of the shear deflection is negligible, and that the assumed cubic law for the warping function gives the same distribution law of the tangential stresses on the branch, as that one obtained by the shear flow theory.

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